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Minimax Linear Predictor  
under Lipschitz' Type Conditions  
for the Regression Function

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# ABSTRACT

Suppose that  $Y_i = f(x_i) + \epsilon_i$ ,  $i = 1, \dots, n$ , where  $\epsilon_i$  are independent random variables with  $E(\epsilon_i) = 0$ ,  $E(\epsilon_i^2) = \sigma^2$ ,  $E(\epsilon_i \epsilon_j) = 0$ ,  $i \neq j$ .

We want to estimate  $f(x_0)$  by a linear function  $\hat{f}(x_0) = \sum_{i=1}^n a_i Y_i$ . Those coefficients  $a_i$  which minimize

$$\sup_f E(\hat{f}(x_0) - f(x_0))^2 \quad \text{for some class of } f$$

are sought. In two cases where

$$(a) \quad \left| \frac{f(x_i) - f(x_j)}{x_i - x_j} \right| \leq c\sigma,$$

$$(b) \quad x_i = i, x_0 = 0$$

$$|\Delta^{h+1} f(k)| \leq c\sigma \quad \text{where } \Delta f(k) = f(k+1) - f(k),$$

the minimax solution is obtained, and it is shown that the solution coincides with weight least squares with weight

$$w_i = \max (\lambda - \mu |x_i - x_j|, 0).$$



## §0. Introduction.

Regression analysis is the most (or at least one of the most) popular and most often used techniques in various fields of statistical data analysis. In some cases, however, regression analysis is very dangerous, and sometimes gives awkward results. Such dangers, which are inherent in regression techniques, are well known, at least well perceived by experienced applied statisticians. But theoretical analysis of such a situation that yields some pitfalls to the careless application of regression analysis is far from satisfactory. Though well trained statisticians can evade such a danger by their good judgment, there is no formal well established technique that may be applied. The purpose of this paper is to derive some method to treat one such difficulty, i.e. the problem of the functional form of the regression.

Suppose that we have a quantity or response  $Y$ , which is influenced by some quantity or explanatory variable  $x$ , and we assume a relation, as

$$Y = f(x) + U ,$$

where  $f(x)$  denotes the influence of  $x$  on  $Y$  and  $U$  is a random variable, which may be the error of measurement, or random shock or something similar. We assume that  $n$  pairs of observations  $(x_i, Y_i)$ ,  $i = 1, \dots, n$ , are given,

and for each pair

$$Y_i = f(x_i) + U_i ; \quad i = 1, \dots, n$$

and  $U_1, \dots, U_n$  are distributed independently and with mean 0 and variance  $\sigma^2$ . Such assumptions on the distribution of  $U$  may well be the source of trouble in many practical situations, but we shall not discuss the problem here, and simply take it for granted that the above assumptions are true.

Our problem is to estimate the form of the function  $f(x)$ . But without any assumption on  $f$ , we can go no further, and it is usually assumed that  $f(x)$  is, or nearly is, a linear function, and it is expressed as

$$Y_i = \alpha + \beta x_i + U_i , \quad i = 1, \dots, n,$$

where  $\alpha$  and  $\beta$  are unknown parameters and are estimated by least squares.

In many cases a linear function gives a reasonably good approximation to the function  $f(x)$ , or random error is so large that any indication of nonlinearity of  $f(x)$  does not show up. But in other cases, and not in rare cases, the data significantly contradict the linearity assumption of the function  $f(x)$ . Then the quadratic function will be used, and if the quadratic function is still unsatisfactory, cubic, or polynomials of higher degree will be used. And if sufficiently high degrees are allowed, the data will be



fitted very closely by a polynomial of the type

$$\hat{f}(x) = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_p x^p.$$

Since any continuous function can be approximated to any precision by a polynomial, the above procedure may seem plausible; for, in practical situations, it can be assumed without any doubt that  $f(x)$  is continuous, and smooth enough. But it also happens in practical situations that a polynomial of higher degree fits the data very nicely, but gives ridiculous values when extrapolated.

The simplest example is the case when  $x$  is the time  $t$ . We have a time series data for some period, for  $t = 1, 2, \dots, T$ , and we would like to predict the value of  $Y$  for the next period, i.e. for  $t = T+1$ . In this case, if no other relevant information is available, we assume a model of the type

$$Y = \beta_0 + \beta_1 t + \dots + \beta_p t^p + u_t,$$

and estimating  $\beta_0 \dots \beta_p$  by least squares, and substituting  $t = T+1$ , we have a predicted value of  $Y$  for the next period. And it often happens that when the data is very long, though a simple function such as a linear or quadratic function fails to be fitted to the data, using a high degree polynomial, we have ridiculous values for the next period. Consequently, it may seem intuitively plausible to use only the latter part

of the data and to fit a simple function such as one that is linear or quadratic in order to predict the near future. Or more sophisticatedly, we can use a system of weight  $w_1 w_2 \dots w_T$ , which is increasing with  $t$  (i.e. decreasing as the data tends to the distant past), and fitting a function by weighted least squares using those weights. And it will be shown actually that such a weight system, i.e. a system for which  $w_t = 0$  for  $t \leq t_0$  and  $w_t = a - bt$ , for  $t > t_0$ , is optimum in some minimax sense.

We shall deal with the problem from the viewpoint of prediction, or estimation of the value of the function at the specific value of  $x = x_0$ . And if  $\hat{f}(x_0)$  is a predictor or estimator, we shall use as a criterion

$$E(\hat{f}(x_0) - f(x_0))^2$$

i.e. the mean square error. And we assume that nothing definite is known about the functional form of  $f(x)$  but that it satisfies some regularity conditions. And we shall seek a predictor which minimizes the supremum of mean square error with respect to the class of functions satisfying those conditions.

§1. Minimax Linear Prediction under the first order Lipschitz condition.

First we shall consider the simplest case.

Let  $(x_i, Y_i)$ ,  $i=1, 2, \dots, n$ , be a pair of data, and assume that

$$Y_i = f(x_i) + U_i$$

and

$$E(U_i) = 0, \quad i=1, 2, \dots, n; \quad E(U_i^2) = \sigma^2, \quad i=1, 2, \dots, n;$$

$$E(U_i U_j) = 0, \quad i \neq j.$$

We want to predict the value of  $f$  at  $x = x_0$ , i.e.  $f(x_0)$ , by a linear combination of  $Y_i$ ,

$$\hat{f}(x_0) = a_1 Y_1 + \dots + a_n Y_n = \sum_k a_k Y_k.$$

Then the mean square error of  $\hat{f}(x_0)$  is given by

$$E\{(\hat{f}(x_0) - f(x_0))^2\} = \left\{ \sum_k a_k f(x_k) - f(x_0) \right\}^2 + \sum_k a_k^2 \sigma^2.$$

We shall consider the following assumption.

Assumption 1. For any  $x_i, x_j$ ,  $x_i \neq x_j$ , there is a constant  $c$  which satisfies

$$\left| \frac{f(x_i) - f(x_j)}{x_i - x_j} \right| \leq cc. \quad (1.1)$$

And we are to minimize the supremum of mean square error with respect to all functions which satisfy the above

assumption.

In order that the supremum be finite, we must have

$$\sum a_k = 1.$$

We assume that  $x_1 \leq x_2 \leq \dots \leq x_j \leq x_0 \leq x_{j+1} \leq \dots \leq x_n$   
and define

$$d_k = x_{k+1} - x_k \quad \text{for } k = 1, 2, \dots, j-1$$

$$d_j = x_0 - x_j$$

$$d_{j+1} = x_{j+1} - x_0$$

$$d_k = x_k - x_{k-1} \quad \text{for } k = j+2, \dots, n,$$

and also define

$$\Delta_k = f(x_{k+1}) - f(x_k) \quad \text{for } k = 1, 2, \dots, j-1$$

$$\Delta_j = f(x_0) - f(x_j)$$

$$\Delta_{j+1} = f(x_{j+1}) - f(x_0)$$

$$\Delta_k = f(x_k) - f(x_{k-1}) \quad \text{for } k = j+2, \dots, n.$$

Then

$$f(x_k) = f(x_0) - \Delta_j - \dots - \Delta_k \quad \text{for } k \leq j$$

$$f(x_k) = f(x_0) + \Delta_{j+1} + \dots + \Delta_k \quad \text{for } k \geq j+1.$$

Hence, if  $\sum a_k = 1$ ,

$$\begin{aligned} \sum a_k f(x_k) - f(x_0) &= - \sum_{k=1}^j a_k \sum_{i=k}^j \Delta_i + \sum_{k=j+1}^n a_k \sum_{i=j+1}^k \Delta_i \\ &= - \sum_{i=1}^j \Delta_i \sum_{k=1}^i a_k + \sum_{i=j+1}^n \Delta_i \sum_{k=1}^i a_k. \end{aligned}$$

Assumption 1 is equivalent to  $|\Delta_k| \leq cd_k \sigma$ ,  $k=1, \dots, n$ , so that under Assumption 1,

$$\sup \left\{ \sum a_k r(x_k) - r(x_0) \right\}^2 \leq c^2 \sigma^2 \left\{ \sum_{i=1}^j d_i \sum_{k=1}^i a_k + \sum_{i=j+1}^n d_i \sum_{k=1}^n a_k \right\}^2.$$

Without loss of generality we can assume that  $\sigma = 1$ , and the problem is to minimize

$$c^2 \left\{ \sum_{i=1}^n d_i |b_i| \right\}^2 + \sum a_k^2 \quad \text{where} \quad b_i = \sum_{k=1}^i a_k$$

or  $\sum_{k=1}^n a_k \quad (1.2)$

under the condition that  $\sum a_k = 1$ .

To minimize (1.2), it will be shown that we must have  $a_k \geq 0$  for all  $k$ .

Suppose that we have one negative element  $a_\ell < 0$ ,  $1 \leq \ell \leq j$ . It is easily shown that  $b_j$  and  $b_{j+1}$  must be nonnegative. Hence either  $b_\ell = \sum_{i=1}^{\ell} a_i \geq 0$  or  $b_\ell < 0$ , and there is one pair  $\ell < h$  such that  $a_\ell < 0$ ,  $a_{\ell+1} = \dots = a_{h-1} = 0$ ,  $a_h > 0$ . Then define a new system of coefficients  $a'_k$  by

$$a'_\ell = 0 \quad \text{and} \quad a'_k = b_\ell a_k / b_{\ell-1}, \quad \text{for } k < \ell;$$

$$= a_k, \quad \text{otherwise};$$

for the first case, and

$$a'_\ell = a_\ell + c, \quad a'_h = a_h - c, \quad \text{where } c = \min(a_\ell, a_h);$$

$$a'_k = a_k \quad \text{otherwise;}$$

for the second case. And in either case we have

$$|a'_k| \leq |a_k| \quad \text{and} \quad |b'_i| \leq |b_i| \quad \text{for all } k, i$$

and

$$|a'_\ell| < |a_\ell|.$$

Hence we have

$$c^2 \left\{ \sum_{i=1}^n d_i |b'_i| \right\}^2 + \sum a_k'^2 < c^2 \left\{ \sum d_i |b_i| \right\}^2 + \sum a_k^2.$$

Thus the problem is reduced to minimizing

$$c^2 \left\{ \sum_{i=1}^j d_i \sum_{k=1}^i a_k + \sum_{i=j+1}^n d_i \sum_{k=1}^n a_k \right\}^2 + \sum a_k^2$$

$$= c^2 \left\{ \sum_{k=1}^j \left( \sum_{i=k}^j d_i \right) a_k + \sum_{k=j+1}^n \left( \sum_{i=j+1}^k d_i \right) a_k \right\}^2 + \sum a_k^2 \quad (1.3)$$

under the condition that  $\sum a_k = 1$  and  $a_k \geq 0$ .

Denoting the Lagrangian multiplier by  $2\lambda$  and differentiating, we have for the condition of minimum,

$$c^2 \delta_k M + a_k - \lambda = 0 \quad \text{if } a_k > 0$$

$$\geq 0 \quad \text{if } a_k = 0 \quad (1.4)$$

$$\text{where } \delta_k = \sum_{i=k}^j d_i = |x_0 - x_k| \quad \text{for } i \leq j$$

$$= \sum_{i=j+1}^k d_i = |x_0 - x_k| \quad \text{for } i \geq j+1;$$

and  $M = \sum \delta_k a_k$ .

or, in other words,

$$\begin{aligned} a_k &= \max (\lambda - c^2 M \delta_k, 0) \\ M &= \sum \delta_k a_k \quad (1.5) \\ \sum a_k &= 1 \end{aligned}$$

There is a unique set of solutions for  $a_k$  and  $\lambda$  which satisfies (1.5), which is calculated as follows. Let the number of the sample be reordered according to the size of  $\delta$ , i.e. let  $0 \leq \delta_1 < \delta_2 < \dots < \delta_n$ . Then  $a_k > 0$  for  $k \leq m$  for some  $m$  and  $a_k = 0$  for  $k \geq m+1$ , and from (1.5)

$$c^2 M \delta_m < \lambda \leq c^2 M \delta_{m+1}$$

and

$$M = \sum_{k=1}^m \delta_k \lambda - c^2 M \sum_{k=1}^m \delta_k^2.$$

Hence

$$\sum_{k=1}^m \delta_k (\delta_m - \delta_k) < \frac{1}{c^2} \leq \sum_{k=1}^{m+1} \delta_k (\delta_{m+1} - \delta_k) = \sum_{k=1}^{m+1} \delta_k (\delta_{m+1} - \delta_k).$$

Define  $R_m = \sum_{k=1}^m \delta_k (\delta_m - \delta_k)$ , then  $R_m$  is monotone increasing and  $m$  is determined from

$$R_m < \frac{1}{c^2} \leq R_{m+1}.$$

Once  $m$  is determined,  $\lambda$  and  $M$  are determined easily from (1.5).

There may be some complexity if some of the  $\delta$ 's are equal, but the argument above applies also in such cases, without any serious modification.

Thus, it is established that the following theorem holds true.

Theorem 1. Under Assumption 1, the coefficients of linear minimax predictor in the sense that it minimizes the supremum of mean square error, are given uniquely by (1.5).

Moreover, the supremum of mean square error for the minimax predictor is given by

$$\lambda = \frac{1 + c^2 \sum_{k=1}^m \delta_k^2}{m(1 + c^2 \sum_{k=1}^m (\delta_k - \bar{\delta})^2)} \quad \text{where } \bar{\delta} = \frac{1}{m} \sum_{k=1}^m \delta_k .$$

The latter half of the theorem is obtained from what follows. The maximum mean square error for given  $a_k$  is equal to

$$c^2 M^2 + \sum a_k^2 .$$

But, for the minimax predictor,

$$c^2 M \delta_k a_k + a_k^2 - \delta a_n = 0 \quad \text{for all } k$$

$$c^2 M \sum \delta_k a_k + \sum a_k^2 = c^2 M^2 + \sum a_k^2 = \lambda \sum a_k = \lambda .$$

And  $\lambda$  is obtained from the equations:



$$\sum a_k = m \lambda = e^{2M} \sum b_k = 1$$

$$\sum b_k a_k = \left( \sum b_k \right) \lambda = e^{2M} \sum b_k = M.$$

### 3.1. Minimax predictor under higher order conditions.

Next we shall consider a more complicated situation, that is we shall consider a higher order Lipschitz condition. On the other hand, we assume simply that  $x_i$  are placed at equal distances, i.e. we shall assume that  $x_i = i$ ,  $i = 1, \dots, n$ , and that  $x_0 = 0$ .

We define a difference operator  $\Delta$  by

$$\Delta f(x) = f(x+1) - f(x)$$

and power  $\Delta^j$  of  $\Delta$  by

$$\Delta^j(f(x)) = \Delta(\Delta^{j-1}f(x+1) - \Delta^{j-1}f(x)), \quad j = 1, 2, \dots$$

We assume that

Assumption 2.  $|\Delta^{h+1}f(x)| \leq c\sigma,$

and we shall obtain a minimax linear predictor under this assumption. For simplicity we put  $\sigma = 1$  as before.

Let  $\hat{f}(0) = \sum a_k Y_k$  be a linear predictor. Then

$$E(\hat{f}(0) - f(0))^2 = \left( \sum a_k f(k) - f(0) \right)^2 + \sum a_k^2,$$

and  $f(k)$  can be expressed as

$$\begin{aligned}
f(k) = & f(0) + {}_k c_1 d_1 + {}_k c_2 d_2 + \dots + {}_k c_k d_k \\
& + \sum_{j=0}^{k-h-1} {}_{k-j-1} c_k \Delta^{h+1} f(j) , \quad k = 1, 2, \dots,
\end{aligned} \tag{2.1}$$

where  $d_i = \Delta^i f(0)$ ,  ${}_k c_i$  is the binomial coefficient with the definition that  ${}_k c_i = 0$  if  $1 \leq k < i$ .

If no restrictions are imposed on  $f(0)$  and  $d_1 \dots d_k$ , we must have

$$\sum_{k=1}^n a_k = 1 , \quad \sum_{k=1}^n {}_k c_j a_k = 0 , \quad j = 1, 2, \dots, k, \tag{2.2}$$

in order that  $\sup E(\hat{f}(0) - f(0))^2 < \infty$ .

And under the assumption above,

$$\sup E(\hat{f}(0) - f(0)) = c^2 \left( \sum_{j=0}^{n-h-1} |b_j| \right)^2 + \sum a_k^2 \tag{2.3}$$

where  $b_j = \sum_{k=j+h+1}^n {}_{k-j-1} c_h a_k$ .

Thus it is necessary to minimize (2.3) under the condition (2.2).

Let the Lagrangian form  $\Phi$  be

$$\begin{aligned}
\Phi(a_1 \dots a_n) = & c^2 \left( \sum_{j=0}^{n-h-1} \left| \sum_{k=j+h+1}^n {}_{k-j-1} c_h a_k \right| \right)^2 + \sum a_k^2 \\
& - 2 \lambda_0 \sum a_k - 2 \lambda_1 \sum {}_k c_1 a_k \dots \\
& - 2 \lambda_h \sum {}_k c_h a_h .
\end{aligned}$$

Since  $\Phi$  is strictly convex in its arguments, a local minimum gives the global minimum, and if for suitable  $\lambda_1 \dots \lambda_n$ , the local minimum satisfies the condition (2.1) it gives the solution of our problem. And a necessary and sufficient condition for local minimality is given by

$$\left( \frac{\partial \Phi}{\partial a_k} \right)_+ \geq 0, \quad \left( \frac{\partial \Phi}{\partial a_k} \right)_- \leq 0, \quad k=1,2,\dots,n,$$

where  $(\partial \Phi / \partial a_k)_+$  and  $(\partial \Phi / \partial a_k)_-$  are the right and the left derivatives respectively, which are given by

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \Phi}{\partial a_k} \right)_+ &= c^{2M} \sum_{j=0}^{k-h-1} k-j-1 c_h \operatorname{sgn}(b_j+0) + a_k - \lambda_0 - \lambda_1 k c_1 \\ &\quad - \dots - \lambda_h k c_h; \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \Phi}{\partial a_k} \right)_- &= c^{2M} \sum_{j=0}^{k-h-1} k-j-1 c_h \operatorname{sgn}(b_j+0) + a_k - \lambda_0 - \lambda_1 k c_1 \\ &\quad - \dots - \lambda_h k c_h, \end{aligned}$$

where  $b_j = \sum_{k=j+h+1}^n k-j-1 c_h a_k$ ,  $M = \sum_{j=0}^{n-h-1} |b_j|$  and

$$\begin{aligned} \operatorname{sgn}(b+0) &= 1 && \text{if } b \geq 0 \\ &= -1 && b < 0 \\ \operatorname{sgn}(b-0) &= 1 && \text{if } b > 0 \\ &= -1 && b \leq 0. \end{aligned}$$

We shall conjecture that the solution will have the form

$$a_k = 0 \text{ for } k = m+1 \dots n \text{ and } a_m \neq 0 \text{ for some } m \quad (2.4)$$

and  $b_j$  are of the same sign for all  $j$ .

If this conjecture were correct, for  $k \leq m$  we have

$$a_k = \beta_0 + \beta_1 k + \dots + \beta_{h+1} k^{h+1} \quad (2.5)$$

since for  $k \leq m$ ,

$$\left(\frac{\partial \phi}{\partial a_k}\right)_+ = \left(\frac{\partial \phi}{\partial a_k}\right)_{-1} = 0 .$$

We shall prove that this conjecture is actually true, which will be done in several steps.

First we shall define the following terminology.

Definition. For a sequence  $a_1 \dots a_m$ , we shall say that it changes sign at  $a_i$ , if

$$a_i a_{i+1} < 0 , \text{ or for some } j \quad a_{i+1} = \dots = a_{i+j-1} = 0$$

$$\text{and} \quad a_i a_{i+j} < 0 .$$

Then we have

(1) If for  $a_1 \dots a_m$ ,  $\sum_{k=1}^m k^{c_j} a_k = 0$  for  $j = 1, \dots, h$ ,  $a_1 \dots a_m$  changes sign at least  $h$  times provided that  $a_k \neq 0$ .

Proof. The condition implies that

$$\sum_{k=1}^m k^{c_j} a_k = 0 , \quad k = 1, \dots, h .$$

Assume that  $a_1, \dots, a_m$  changes sign at  $a_{i_1}, \dots, a_{i_r}$ , and define a polynomial

$$g(t) = t(i_1 + \frac{1}{t} - t) \dots (i_r + \frac{1}{t} - t) .$$

If  $r \geq h-1$ , we have from the assumption

$$\sum g(k) a_k = 0 .$$

But by the definition of  $g$ , for all  $k$  we must have

$$\begin{aligned} g(k)a_k &\geq 0 \quad \text{if the first non-zero element} > 0 \\ &\leq 0 \quad \text{if it is} \leq 0, \end{aligned}$$

which implies that  $a_k \equiv 0$  .

Q.E.D.

(2) For  $a_k$  satisfying the condition of the solution as well as (2.4) and (2.5) we must have

$$a_k = \beta(\alpha_1 - k) \dots (\alpha_{h+1} - k) \quad (2.6)$$

where  $1 \leq \alpha_1 < \dots < \alpha_h < m < \alpha_{h+1}$  and  $\beta > 0$ , and  $\alpha_{h+1} \leq m+1$  if  $m < n$ .

Proof. Since  $b_{n-h-1} = {}_m J_h a_m$ ,

$$\operatorname{sgn} |b_j| = \operatorname{sgn} |a_m|$$

if all  $b_j$  have the same sign. Hence

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \Phi}{\partial a_m} \right)_+ &= \frac{1}{2} \left( \frac{\partial \Phi}{\partial a_m} \right)_- = c^{2M} {}_m c_{h+1} \operatorname{sgn} |a_m| + a_m - \lambda_0 - \lambda_1 {}_m c_1 - \dots \\ &\quad - \lambda_h {}_m c_h = 0 \end{aligned}$$

implies that the signs of  $\beta_{h+1}$  in (2.5) and  $a_m$  are opposite.

By (1) it is shown that the equation

$$\beta_0 + \beta_1 x + \dots + \beta_{k+1} x^h = 0$$

must have at least h distinct roots in the interval (1,m), and the above implies that it has one which is larger than m.

And if  $\beta_{h+1} > 0$ , then  $a_{m+1} = 0$  for the solution implies

$$\left(\frac{\partial \phi}{\partial a_{m+1}}\right)_+ = (-1)^{h+1} \beta(\alpha_1 - (m+1)) \dots (\alpha_{h+1} - (m+1)) \geq 0 ;$$

hence  $m+1 \geq \alpha_{h+1}$ .

Similarly for  $\beta_{h+1} < 0$ ,  $(\partial \phi / \partial a_{m+1})_- \leq 0$  implies  $m+1 \geq \alpha_{h+1}$ .

And

$$\begin{aligned} \sum_k (\alpha_1 - k) \dots (\alpha_h - k) a_k &= \sum \alpha_1 \dots \alpha_h a_k = \alpha_1 \dots \alpha_h \\ &= \sum \beta(\alpha_1 - k)^2 \dots (\alpha_h - k)^2 (\alpha_{h+1} - k), \end{aligned}$$

implies that  $\beta > 0$ .

Q.E.D.

$$(3) \quad \begin{aligned} \text{If } a_k &= \beta(\alpha_1 - k) \dots (\alpha_h - k)(\alpha_{h+1} - k) \quad \text{for } k \leq m \\ &= 0 \quad \text{for } k \geq m+1 \end{aligned}$$

and  $\beta > 0$ ,  $1 < \alpha_1 < \dots < \alpha_h < m < \alpha_{h+1}$ ,

and it holds that

$$\sum k^{c_j} a_k = 0, \quad j = 1, \dots, h.$$

Then  $(-1)^n b_j = (-1)^n \sum_{k=j+n+1}^m (k-j-1) g_{n,j} a_k = 0$  for  $j = 1, \dots, m-n-1$ .

Define a function

$$\begin{aligned} g_{h,j}(x) &= (x-j-1) \dots (x-j-h)/h! , & x \geq j+n, \\ &= 0 , & x \leq j+n. \end{aligned}$$

We shall show that for any polynomial  $h(x)$  of degree  $h$ , a sequence  $\{c_k\}$  defined by  $c_k = h(k) - g_{h,j}(k)$ ,  $k=1,2,\dots$  cannot change the sign more than  $h+1$  times.

If  $h = 1$ , this is obvious.

For  $h \geq 2$ , define a new sequence  $\{c'_k\}$  by

$$c'_k = \Delta c_k = c_{k+1} - c_k , \quad k = 1, 2, \dots,$$

then  $c'_k = h'(k) - g_{h-1,j}(k)$ , where  $h'(k)$  is a polynomial of degree  $h-1$ , and if  $\{c_k\}$  changes sign more than  $h+1$  times,  $\{c'_k\}$  must change sign more than  $h$  times.

Hence, by induction, the proposition is proved.

Define a polynomial  $h^*(x) = \gamma_1 x + \dots + \gamma_n x^n$  such that

$$h^*(\alpha_j) = g_{h,j}(\alpha_j) \quad \text{for } j=1, \dots, h.$$

Then the equation  $h^*(x) - g_{h,j}(x) = 0$  has  $h+1$  distinct roots,  $0, \alpha_1, \dots, \alpha_n$ , and no other roots than that. Hence  $\gamma_n > 0$ , and since  $1 < \alpha_1$ ,  $(-1)^h (h^*(1) - g_{h,j}(1)) < 0$ . Thus,

$$(-1)^h (h^*(k) - g_{h,j}(k)) a_k \leq 0 \quad \text{for all } k=1, 2, \dots, m,$$

the equality being satisfied only when  $a_k = 0$ .

Hence

$$\begin{aligned} 0 &> \sum (-1)^h (h^*(k) - g_{h,j}(k)) a_k \\ &= (-1)^h \sum h^*(k) a_k - (-1)^h \sum g_{h,j}(k) a_k \\ &= - (-1)^h b_j, \end{aligned}$$

the equality being omitted since  $a_m \neq 0$ . Q.E.D.

(4) For any  $c^2 > 0$ , there is an  $\alpha^*$  and a polynomial

$$\psi(k) = \gamma_0 + \gamma_1 k + \dots + \gamma_h k^h$$

such that

$$\begin{aligned} \sum_{k=1}^m (\alpha^* - k) \psi(k) &= 1 \\ \sum_{k=1}^m m c_j (\alpha^* - k) \psi(k) &= 0, \quad j = 1, 2, \dots, h, \\ \sum_{k=1}^m m c_{h+1} (\alpha^* - k) \psi(k) &= (-1)^h \gamma_h / c^2, \end{aligned}$$

where  $m = \min(n, [\alpha^*])$ .

Proof. First we shall show that for fixed  $\alpha \geq m+1$ , there is a polynomial

$$\bar{\psi}_\alpha(k) = \bar{\gamma}_0 + \bar{\gamma}_1 k + \dots + k^h,$$

for which



$$\sum_{k=1}^m k^{c_j} (\alpha - k) \bar{\psi}_\alpha(k) = 0, \quad j = 1, \dots, m, \quad (2.6)$$

where  $m = \min(n, [\alpha])$ , if  $n \geq m+1$ .

We shall express  $\bar{\psi}_\alpha(k)$  in the form

$$\bar{\psi}_\alpha(k) = \lambda'_0 + \lambda'_1 k^{c_1} + \dots + \lambda'_{h-1} k^{c_{h-1}} + k^{c_h}.$$

Then putting  $w_k = \sqrt{\alpha - k}$ , equation (2.6) can be expressed as

$$\sum_{j'=0}^{h-1} \left( \sum_{k=1}^m (w_k k^{c_j})(w_k k^{c_{j'}}) \right) \lambda'_j = - \sum_{k=1}^m (w_k k^{c_j})(w_k k^{c_h}),$$

$$j' = 0 \dots h-1, \quad (2.7)$$

Since as functions of  $k$ ,  $w_k k^{c_j}$  are linearly independent if  $m \geq h+1$ , the coefficient matrix of  $\lambda'$  in (2.7) is non-singular, and there is a unique solution in  $\lambda'_1 \dots \lambda'_h$ .

Next we shall show that

$$\ell(\alpha) = \sum_{k=1}^m k^{c_{h+1}} \bar{\psi}_\alpha(k) (\alpha - k)$$

is a monotone increasing function of  $\alpha$ .

Suppose that for  $\alpha_1 < \alpha_2$ ,  $m_{\alpha_1} = m_{\alpha_2} = m \geq h+1$ . Then  $\ell(\alpha_2) - \ell(\alpha_1) = \sum_{k=1}^m k^{c_{h+1}} \psi^*(k)$ , where  $\psi^*(k)$  is a polynomial of degree  $h$ , and we have

$$\sum_{k=1}^m k^{c_j} \psi^*(k) = 0, \quad \text{for } j = 1, \dots, h.$$

By (1),  $\{\psi^*(k)\}$ ,  $k = 1, 2, \dots$ , changes sign  $h$  times, and

hence it has the form

$$\psi^*(k) = \beta'(\alpha_1' - k) \dots (\alpha_h' - k)$$

where  $1 < \alpha_1' < \dots < \alpha_h' < m$ . Substituting  $k = \alpha_1$

$$\psi^*(\alpha_1) = \bar{\psi}_{\alpha_2}(\alpha_1) (\alpha_2 - \alpha_1) .$$

Since  $\bar{\psi}_{\alpha_2}(k) = 0$  has  $h$  roots all smaller than  $m$ ,  $\bar{\psi}_{\alpha_2}(\alpha_1) > 0$ , hence

$$\psi^*(\alpha_1) > 0 , \quad (-1)^h \beta' > 0 .$$

Since

$$\sum k^j \psi^*(k) = 0 , \quad j = 1, 2, \dots, h,$$

$$\begin{aligned} \sum k^{c_{h+1}} \psi^*(k) &= \sum k^{h+1} \psi^*(k) \\ &= \sum k(k - \alpha_1') \dots (k - \alpha_h') \psi^*(k) \\ &= (-1)^h \beta' \sum k(k - \alpha_1')^2 \dots (k - \alpha_h')^2 > 0. \end{aligned}$$

Thus it is shown that  $\ell(\alpha)$  is monotone increasing in each interval  $m \leq \alpha \leq m+1$ , where  $m$  is an integer, and since  $\ell(h+1) = 0$ , and  $\ell(\alpha) \rightarrow 0$  as  $\alpha \rightarrow \infty$ ,  $\ell(\alpha)$  is monotone increasing from 0 to infinity, as  $\alpha$  increases

from  $\alpha = h+1$ . Hence there is a unique solution that  $\ell(\alpha^*) = 1/c^2$ . And it is also easily shown that

$\sum (\alpha - k) \bar{\psi}_{\alpha}^*(k) \neq 0$ , and by defining  $\gamma_h$  by  $\gamma_h = (\sum (\alpha - k) \bar{\psi}_{\alpha}(k))^{-1}$ ,

and putting  $\psi(k) = \gamma_{h+1}^* - \gamma_{h+1} = (k)$  we have the desired solution. Q.E.D.

Now we can prove the main theorem of this section.

Theorem 2. Let  $\alpha^*$  and  $\psi(k)$  be defined as in (4) above.

Then

$$\begin{aligned} a_k &= (\alpha^* - k)\psi(k) \quad \text{for } k \leq m, \\ &= 0 \quad \text{for } k \geq m+1, \end{aligned}$$

gives the coefficients of the minimax linear predictor under Assumption 2.

Proof. From (1) and (3), for such  $a_k$ ,  $(-1)^{h_{b_j}} = \sum_{k=j-1}^{\infty} c_h a_k$   
 $> 0$  for  $j = 0, 1, \dots$

Hence

$$\left(\frac{\partial \phi}{\partial a_k}\right)_+ = \left(\frac{\partial \phi}{\partial a_k}\right)_- = 0 \quad \text{for } k=1, \dots, m.$$

And, if  $m+1 \leq \alpha^*$ ,

$$\frac{1}{2} \left(\frac{\partial \phi}{\partial a_k}\right)_+ = -(\alpha^* - k)\psi(k) \geq 0 \quad \text{if } h \text{ is even}$$

$$\frac{1}{2} \left(\frac{\partial \phi}{\partial a_k}\right)_- = -(\alpha^* - k)\psi(k) \leq 0 \quad \text{if } h \text{ is odd,}$$

for  $k \geq m+1$ .

Put for even  $h$

$$\begin{aligned} -(\alpha^* - k)\psi(k) &= \mu_k c_{h+1} + \lambda_0 + \lambda_1 c_1 + \dots + \lambda_h k c_h \\ &= \mu_k c_{h+1} + \psi_1(k). \end{aligned}$$

Then  $\mu > 0$  and  $\psi_1(k)$  is a polynomial of degree  $h$ .

We shall show that if the sequence  $a_k$ ,  $k=1,2,\dots$ , defined by  $a_k = \mu_k c_{h+1} + \psi_1(k)$  where  $\psi_1(k)$  is a polynomial of degree,  $h$  changes sign  $h+1$  times, and suppose that  $a_m$  is the term at which the  $h+1^{\text{th}}$  change occurs, then  $\psi_1(k)$  is negative and monotone decreasing for  $k \geq m$ . The proof will be done by induction.

When  $h = 0$ ,  $\psi_1(k)$  being a constant it is obvious.

When  $h \geq 1$ , let a new sequence be defined by

$$a'_k = \Delta a_k = a_{k+1} - a_k = \mu_k c_h + \psi'_1(k),$$

and  $\psi'_1(k)$  is a polynomial of degree  $h-1$ , and defined by

$$\psi'_1(k) = \psi(k+1) - \psi(k).$$

Then  $\{a'_k\}$  changes sign  $h$  times, and the  $h$ -th change occurs not after  $a'_m$ . Thus if the proposition were true for  $h-1$ , then

$$\psi'_1(k) < 0 \text{ for all } k \geq m.$$

And since  $a_m = \mu_m c_{h+1} + \psi(m) < 0$ , we have  $\psi(m) < 0$  and  $\psi(k) < 0$  for all  $k \geq m$ .

By this proposition with even  $h$ , it is shown that

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \phi}{\partial a_k} \right) \Big|_{a_k=0} &= \mu_m c_{h+1} - \mu \sum_{\ell=m+1}^k \ell c_h + \lambda_0 + \lambda_1 k c_1 \\ &+ \dots + \lambda_k k c_h \end{aligned}$$

$$\begin{aligned}
&= \mu_m c_{h+1} + \psi_1(k) \\
&= \mu_m c_{h+1} + \psi_1(m) + (\psi_1(k) - \psi_1(m)) \\
&= a_m + (\psi_1(k) - \psi_1(m)) < 0 \quad \text{for } k \geq m+1.
\end{aligned}$$

Quite similarly for odd  $h$  it is shown that

$$\frac{1}{2} \left( \frac{\partial \phi}{\partial a_k} \right)_{+} \Big|_{a_k=0} > 0 \quad \text{for } k \geq m+1.$$

Which completes the proof.

Q.E.D.

It should be noted that the solution given by Theorem 2 above coincides with fitting polynomials of degree  $h$  by weighted least squares, with weight  $w_k = \alpha^* - k$  for  $k < \alpha^*$  and  $w_k = 0$  for  $k \geq \alpha^*$ .

To see this, suppose that

$$\hat{f}(x) = \hat{\beta}_0 + \hat{\beta}_1 x + \dots + \hat{\beta}_h x^h,$$

be the weighted least squares estimator for  $\hat{f}(x)$ .

Then  $f(0) = \beta_0$  will be a linear combination of  $Y$  of the form

$$\begin{aligned}
\hat{\beta}_0 &= \lambda_0 \sum_{k=1}^n w_k Y_k + \lambda_1 \sum_{k=1}^n w_k k Y_k + \dots + \lambda_h \sum_{k=1}^n w_k k^h Y_k \\
&= \sum (\alpha^* - k) (\lambda_0 + \lambda_1 k + \dots + \lambda_h k^h) Y_k \\
&= \sum a_k Y_k.
\end{aligned}$$

And  $a_k$  must satisfy

$$\sum a_k = 1, \quad \sum k^c_j a_k = 0, \quad j=1, \dots, h.$$

Thus  $a_k$  must be equal to those previously obtained.

Arithmetically diminishing weights may seem intuitively appealing, and in fact it was shown to be optimum in the above sense, such a procedure may be recommended in practical situations.

### §3. Some Numerical Discussion.

We shall briefly discuss numerical properities of the results of the previous section.

If  $\alpha^*$  defined in the previous section is equal to  $m+1$ ,  $a_k$  is obtained from the relation that

$$a_k = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \dots + \alpha_{h+1} k^{h+1} = \phi_h(k) \quad \text{for } k \leq m$$

and

$$\phi_h(m+1) = 0,$$

$$\sum_{k=1}^m \phi_h(k) = 1,$$

$$\sum_{k=1}^m k^c_j \phi_h(k) = 0, \quad j=1, \dots, h.$$

And it is shown after some algebraic calculations that

$$\begin{aligned}
\phi_0(k) &= \frac{1}{m(m+1)} (m+1-k) \\
\phi_1(k) &= \frac{1}{m(m+1)(m-1)} (m+1-k)(m+1-2k) \\
\phi_2(k) &= \frac{12}{m(m+1)(m-1)(m-2)} (m+1-k)(m^2+2m+2-5(m+1)k+5k^2) \\
\phi_3(k) &= \frac{20}{m(m+1)(m-1)(m-2)(m-3)} (m+1-k)(m+1-2k) \\
&\quad \times (m^2+2m+6-7(m+1)k+7k^2).
\end{aligned} \tag{3.1}$$

And these functions give the coefficients of minimax predictors corresponding to the case when

$$\begin{aligned}
c^2 &= \text{absolute value of the coefficient of the} \\
&\text{highest power in } \phi_h(k) = \sum_k c_{h+1} \phi(k)
\end{aligned}$$

or when

$$c^2 = c_m^2 = \frac{K_h}{m+h+2} c_{2h+2} \tag{3.2}$$

where  $K_0 = 1$ ,  $K_1 = 4$ ,  $K_2 = 15$ ,  $K_3 = 56$ , etc.

And the supremum of the mean square error is simply equal to  $\phi_h(0)$ , which will be chosen as follows.

It was shown that

$$a_k = \phi_h(k) = \lambda_0 + \lambda_1 k^{c_1} + \dots + \lambda_h k^{c_h} + \lambda_{h+1} k^{c_{h+1}}$$

$$\text{and } \lambda_{h+1} = -c^2 M, \quad M = \sum_k c_{h+1} a_k.$$

And the mean square error is given

$$\begin{aligned}
 c^2 M^2 + \sum_k a_k^2 &= c^2 M \sum_k c_{h+1} a_k + \sum_k a_k^2 \\
 &= \sum_k a_k (c^2 M c_{h+1} + a_k) \\
 &= \sum_k a_k (\lambda_0 + \lambda_1 c_1 + \dots + \lambda_h c_h) = \lambda_0 = \phi_h(0) .
 \end{aligned}$$

And from (3.1) it is obtained

$$\begin{aligned}
 \phi_0(0) &= \frac{2}{m} \\
 \phi_1(0) &= \frac{6(m+1)}{m(m-1)} \\
 \phi_2(0) &= \frac{12(m^2+2m+2)}{m(m-1)(m-2)} \\
 \phi_3(0) &= \frac{20(m+1)(m^2+2m+6)}{m(m-1)(m-2)(m-3)}
 \end{aligned} \tag{3.3}$$

When  $c^2$  is given, we should choose  $m$  such that

$$c_m^2 \geq c^2 > c_{m+1}^2$$

and  $a_k = h(k)$  may be obtained in the way indicated in the previous section.

We shall show some of the numerical values below.



$c_m^2$ 

m	1	2	3	4	5	6	7	8	9	10
h										
0	1	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{56}$	$\frac{1}{84}$	$\frac{1}{120}$	$\frac{1}{165}$	$\frac{1}{220}$
1	$\infty$	4	$\frac{2}{3}$	$\frac{4}{21}$	$\frac{1}{14}$	$\frac{2}{63}$	$\frac{1}{63}$	$\frac{2}{231}$	$\frac{1}{198}$	$\frac{4}{1287}$
2		$\infty$	15	$\frac{15}{8}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{3}{22}$	$\frac{5}{88}$	$\frac{15}{572}$	$\frac{15}{1144}$
3			$\infty$	56	$\frac{28}{5}$	$\frac{56}{55}$	$\frac{14}{55}$	$\frac{56}{715}$	$\frac{4}{143}$	$\frac{8}{715}$

 $\phi_h(0)$ 

m	1	2	3	4	5	6	7	8	9	10
h										
0	2	1	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$
1	$\infty$	9	4	$\frac{5}{2}$	$\frac{9}{5}$	$\frac{7}{5}$	$\frac{8}{7}$	$\frac{27}{28}$	$\frac{5}{6}$	$\frac{11}{15}$
2		$\infty$	34	13	$\frac{37}{5}$	5	$\frac{26}{7}$	$\frac{41}{14}$	$\frac{101}{42}$	$\frac{61}{30}$
3			$\infty$	125	41	21	$\frac{42}{7}$	$\frac{129}{14}$	$\frac{125}{18}$	$\frac{11}{2}$

The value of the supremum of mean square error for intermediate values of  $c^2$  is troublesome to calculate, but rough idea of it will be obtained by interpolation. We shall denote by  $e^2$  the maximum mean square error, and have an approximate formula about the relation between  $e^2$  and  $c^2$ .

When  $h = 0$

$$c_m^2 = \frac{6}{m(m+1)(m+2)} \div \frac{6}{(m+1)^2},$$

and

$$e^2 = \frac{2}{m} \quad (3.4)$$

for integral values of  $m$ , hence we can have approximately,

$$e^2 \doteq \frac{2}{(6/c^2)^{1/3}-1} \doteq \left(\frac{2}{3}\right)^{1/3} c^{2/3} \quad \text{for small } c.$$

Similarly for  $h = 1$

$$e^2 \doteq \frac{6}{(480/c^2)^{1/5}-3} \doteq \left(\frac{81}{5}\right)^{1/5} c^{2/5},$$

and for  $h = 2$ ,

$$e^2 \doteq \frac{12}{(15 \times 7! / c^2)^{1/7}-6} \doteq \left(\frac{2^{10} 3^4}{175}\right)^{1/7} c^{2/7}; \quad (3.4)'$$

for  $h = 3$ ,

$$e^2 \doteq \frac{20}{(56 \times 9! / c^2)^{1/9}-10} \doteq \left(\frac{10^8}{3^4 7^2}\right)^{1/9} c^{2/9}.$$

Generally, for small  $c$ , we have

$$e^2 \doteq \eta_h c^{2/(2h+1)},$$

where  $\eta_h$  is a constant, and seems to be slowly increasing with  $h$ .

In practical situations we may not usually be quite sure about  $h$ , and also about  $c^2$ , so that the above discussions may give us some indications about the choice of  $h$ . And

usually it will be safe to assume rather large value for  $c^2$ , and if we take  $c^2$  not to be very small,  $m$  must be small, and  $e^2$  increases rapidly with  $h$ . As a rough conclusion, we say that it is safer to take rather small  $n$ , and not too large  $m$ .

As an example, suppose that the true regression function is something like

$$f(k) = \alpha \beta^k$$

and we assume that  $\alpha > 0$  and  $\beta < 1$ . (It should be remembered that in our discussion the order of the time sequence is reversed for the next period is denoted by  $k = 0$ .)

Then

$$|\Delta^h f(k)| \leq \alpha(1 - \beta)^h, \quad h = 1, 2, \dots$$

If we are quite sure that  $\alpha \leq 10$ ,  $\beta \geq 0.8$ , then we have the following values for  $c^2$ ,

$$c^2 = 100 \times (0.2)^{2h}.$$

And we have actually

$$\begin{array}{ll} m = 1, & \text{for } h = 0 \\ m = 2, & h = 1 \\ m = 7, & h = 2 \\ m = 9, & h = 3. \end{array}$$

And approximate values for  $e^2$  are given by,

$$\begin{aligned} e^2 &= 5, & \text{for } h &= 0 \\ e^2 &= 4, & h &= 1 \\ e^2 &= 3.9, & h &= 2 \\ e^2 &= 9, & h &= 3. \end{aligned}$$

Hence, in this case the best predictor will fit a quadratic function using 7 values. And in this case  $c^2 = 0.16$  may be replaced by  $c_u^2 = 3/22 = 0.14$ , and, using this, we have from (3.5) the weight system, from (3.1),

$$\frac{21}{14}, \frac{3}{14}, -\frac{5}{14}, -\frac{6}{14}, -\frac{3}{14}, \frac{1}{14}, \frac{3}{14}.$$

And if actually  $f(k) = 10 \times 0.8^k$ , the mean square error of the predictor with the above weight is shown to be equal to 3.56.

Usually,  $c^2$  does not decrease so rapidly as in the exponential case, so that we may have stronger reason to use a rather small value of  $h$ .

#### §4. Application to the Smoothing of Time Series.

The results thus far obtained can be applied to interpolation or smoothing of time series, i.e. in case  $x_0$  is equal to some  $x_i$ .

We shall discuss in this section only the simplest case. That is, we assume that  $x_i = i$ ,  $i = 1, \dots, T$ , and

the Assumption 1 in §1 is taken into consideration.

In this case, the whole discussion in §1 can be applied, and we have the following results.

Suppose that  $x_0 = t$ ,  $1 \leq t \leq T$ . Define  $\tau$  by

$$\tau = \min (t-1, T-t) . \quad (4.1)$$

Then using the notation of §1, we have

$$\begin{aligned} \delta_1 = 0, \delta_2 = \delta_3 = 1, \dots, \delta_{2\tau} = \delta_{2\tau+1} = \tau, \delta_{2\tau+2} = \tau+1, \\ \dots, \delta_T = T - \tau - 1 . \end{aligned}$$

We define  $R_k$  by

$$\begin{aligned} R_1 = 0, R_{2k} = R_{2k+1} = \sum_{i=1}^{2k+1} \delta_i (\delta_{2k+1} - \delta_i) \quad \text{for } k \leq \tau , \\ R'_k = \sum_{i=1}^{k'} \delta_i (\delta_k - \delta_i) , \quad \text{for } k' \geq \tau . \end{aligned}$$

We determine  $m$  to be the largest integer for which  $R_m$  is smaller than  $1/c^2$ . Then we have exactly  $m$  of  $a_k$ 's with positive values. The rest is a matter of simple numerical computation.

In this way, we can have a consistent way of interpolation for all  $t = 1, \dots, T$ , provided that  $c^2$  is the same for all the cases, and we can also extrapolate consistently.

The following are numerical examples.

Assume that  $c^2 = 1/2$ ; then it is shown that if  $\tau \geq 1$ ,

$$a_{t-1} = \frac{1}{4} , a_t = \frac{1}{2} , a_{t+1} = \frac{1}{4} ,$$

and for  $t = 1$ , we have with the same value for  $c^2$ ,

$$a_1 = \frac{7}{12} , a_2 = \frac{1}{3} , a_3 = \frac{1}{12} , \quad \text{for } t = 1,$$

$$a_T = \frac{7}{12} , a_{T-1} = \frac{1}{3} , a_{T-2} = \frac{1}{12} , \quad \text{for } t = T.$$

Similarly for  $c^2 = 1/8$ ,

$$a_{t-2} = \frac{1}{9} , a_{t-1} = \frac{2}{9} , a_t = \frac{1}{3} , a_{t+1} = \frac{2}{9} , a_{t+2} = \frac{1}{9} , \text{ for } t \geq 2,$$

$$a_1 = \frac{16}{66} , a_2 = \frac{23}{66} , a_3 = \frac{16}{66} , a_4 = \frac{9}{66} , a_5 = \frac{2}{66} , \text{ for } t = 2,$$

$$a_1 = \frac{11}{26} , a_2 = \frac{8}{26} , a_3 = \frac{5}{26} , a_4 = \frac{2}{26} , \quad \text{for } t = 1,$$

and similarly for  $t = T$  and  $T-1$ .

For  $c^2 = 1/20$ ,

$$a_t = \frac{1}{4} , a_{t-1} = a_{t+1} = \frac{3}{16} , a_{t-2} = a_{t+2} = \frac{1}{8} , a_{t-3} = a_{t+3} = \frac{1}{16} ,$$

for  $t \geq 3$  ,

$$a_1 = \frac{29}{216} , a_2 = \frac{42}{216} , a_3 = \frac{55}{216} , a_4 = \frac{42}{216} , a_5 = \frac{29}{216} ,$$

$$a_6 = \frac{16}{216} , a_7 = \frac{3}{216} , \quad \text{for } t = 3 ,$$

$$a_1 = \frac{40}{185} , a_2 = \frac{51}{185} , a_3 = \frac{40}{185} , a_4 = \frac{29}{185} , a_5 = \frac{18}{185} ,$$

$$a_6 = \frac{7}{185} , \quad \text{for } t = 2 ,$$

$$a_1 = \frac{5}{15} , a_2 = \frac{4}{15} , a_3 = \frac{3}{15} , a_4 = \frac{2}{15} , a_5 = \frac{1}{15} , \text{ for } t=1.$$

And so on.

Such technique may be applicable to "smoothing" of time series data, and it has some appealing property when compared with the traditional moving average method in that it has diminishing weights as the distance increases, and that it can also produce the smoothed value for both ends of the series coherently.

It will be desirable to extend this technique to the case under more general assumptions. It will seem to be necessary at least to extend to the case for  $h = 1$  of Assumption 2 in §2, i.e. to have predictor or estimator free of the bias due to linear trend. Of course, it is possible to work out for individual case numerically, but unfortunately there is no systematic way of finding a solution in the general cases.

We shall investigate the situation into some detail for  $h = 1$ . From (2.5) we have

$$f(k) = f(0) + kd + \sum_{j=1}^{k-1} (k-j)\Delta^2 f(j+1), \quad k = 1, 2, \dots, \quad (5.2)$$

where slight and obvious modification of notation is introduced.

Suppose that we would like to estimate  $f(t)$ . Then under Assumption 2, for  $h = 1$ , we must have

$$\begin{aligned} \sum a_k &= 1 \\ \sum ka_k &= t. \end{aligned} \quad (5.3)$$

And under this condition, for  $\hat{f}(t) = \sum a_k Y_k$ ,

$$\begin{aligned}
\sup_f |E(\hat{f}(t) - f(t))| &= \sup_{\Delta} \left| \sum_{j=1}^{t-1} (t-j) \Delta_{j+1}^2 - \sum_{k=1}^T \sum_{j=1}^{k-1} (k-j) a_k \Delta_{j+1}^2 \right| \\
&= \sup_{\Delta} \left| \sum_{j=1}^{t-1} \left\{ (t-j) - \sum_{k=j+1}^T (k-j) a_k \right\} \Delta_{j+1}^2 \right. \\
&\quad \left. - \sum_{j=t}^{T-1} \sum_{k=j+1}^T (k-j) a_k \Delta_{j+1}^2 \right| \\
&= \sup_{\Delta} \left| \sum_{j=1}^{t-1} \left( \sum_{k=1}^j (k-j) a_k \right) \Delta_{j+1}^2 - \sum_{j=t}^{T-1} \left( \sum_{k=j+1}^T (k-j) a_k \right) \Delta_{j+1}^2 \right| \\
&= c \left\{ \sum_{j=1}^{t-1} \left| \sum_{k=1}^j (j-k) a_k \right| + \sum_{j=t}^{T-1} \left( \sum_{k=j+1}^T (k-j) a_k \right) \right\}, \quad (5.4)
\end{aligned}$$

where  $\Delta_{j+1}^2 = \Delta^2 f(j+1)$ , for simplicity. We shall denote

$$\begin{aligned}
b_j &= \sum_{k=1}^j (j-k) a_k, & \text{for } j \leq t-1, \\
&= \sum_{k=j}^T (k-j) a_k, & \text{for } j \geq t+1,
\end{aligned}$$

and

$$M = \sum_{j=1}^{t-1} |b_j| + \sum_{j=t+1}^T |b_j|.$$

Then,

$$\sup E(\hat{f}(t) - f(t))^2 = c^2 M^2 + \sum a_k^2.$$

We have to minimize this value under the constraint (5.3). Similarly with §2, we denote by  $\Phi$  the Lagrangian form, and we have



$$\frac{1}{2} \left( \frac{\partial \phi}{\partial a_k} \right)_{\pm} = c^{2M} \sum_{j=k+1}^{t-1} (j-k) \operatorname{sgn}(b_j \pm 0) + a_k - \lambda_0 - k\lambda_1 ,$$

$$\text{for } k \leq t-1 ,$$

$$= c^{2M} \sum_{j=t+1}^{k-1} (k-j) \operatorname{sgn}(b_j \pm 0) + a_k - \lambda_0 - k\lambda_1 ,$$

$$\text{for } k \geq t+1 ,$$

$$= a_k + \lambda_0 - k\lambda_1 , \quad \text{for } k = t . \quad (5.4)$$

We can surmise that if  $t$  and  $T-t$  are both large,  $a_k \geq 0$  for all  $k$ , and  $a_{t+s} = a_{t-s}$ ,  $s = 1, \dots$ , and  $a_{t+s} = a_{t-s} = 0$ , for  $s \geq m+1$ , for some integer  $m$ .

If this conjecture were true, we would have  $b_j \geq 0$  for all  $j$ , and  $= 0$ , for  $j \leq t-m$ , and  $j \geq t+m$ , and we can also suppose that  $\lambda_1 = 0$  from the symmetry. Then (5.4) is reduced to

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \phi}{\partial a_k} \right)_{\pm} &= c^{2M} \frac{|t-k|(|t-k|+1)}{2} + a_k - \lambda_0 , & \text{for } |t-k| \leq m-1, \\ &= \pm c^{2M} \frac{|t-k|(|t-k|+1)}{2} - \lambda_0 , & \text{for } |t-k| \geq m . \end{aligned}$$

Hence if for

$$a_k = \max \left\{ \lambda_1 - \frac{c^{2M}}{2} |t-k|(|t-k|+1), 0 \right\} ,$$

it holds that

$$\sum a_k = 1 \quad \text{and} \quad \sum k a_k = t ,$$

and

$$M = \sum_k \frac{|t-k| (|t-k|+1)}{2} a_k .$$

Then this gives the solution to the problem. The relation between  $m$  and  $c^2$  is obtained from

$$\lambda_0 - \frac{c^2 M}{2} m(m-1) > 0 \geq \lambda_0 - \frac{c^2 M}{2} m(m+1) ,$$

and

$$M = 2 \left( \sum_{j=1}^{m-1} \frac{j(j+1)}{2} - \sum_{j=1}^{m-1} \frac{j^2(j+1)^2}{4} c^2 M \right) ,$$

and it is shown that

$$\frac{m(m-1)(m-2)(m+1)(2m-1)}{30} < \frac{1}{c^2} \leq \frac{m(m-1)(m+1)(m+2)(2m+1)}{30} . \quad (5.5)$$

Consequently, if  $m$  determined from (5.5) is not larger than  $\min(t-1, T-t)$ ,  $a_k$  obtained above gives the solution, thus we have the minimax estimator for intermediate values.

Next we shall consider the case for  $t = 1$ . Then it is shown that

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial \phi}{\partial a_k} \right)_{\pm} &= c^2 M \sum_{j=2}^{k-1} (k-j) \operatorname{sgn}(b_j \pm 0) + a_k - \lambda_0 - k\lambda_1 \text{ for } k \geq 2, \\ &= a_k + \lambda_0 - k\lambda_1 , \quad \text{for } k = 2. \end{aligned}$$

And in this case we have

$$\sum k a_k = 1 \quad \text{or} \quad \sum_{k=2}^T (k-1) a_k = 0 .$$

Similar discussion as in §2 for the case  $t = 0$ , we have also in this case that  $b_j \leq 0$ ,  $j = 2, \dots$ , and

$$\begin{aligned} a_k &= \lambda_0 - k\lambda_1 - \frac{c^2_M}{2} k(k-1) , & \text{for } k \leq m , \\ &= 0 , & k = m , \end{aligned}$$

and  $m$  is determined so that  $a_m < 0$  and

$$\lambda_0 - (m+1)\lambda_1 - \frac{c^2_M}{2} m(m+1) \geq 0 .$$

Thus it should be remarked that the coefficients of the minimax predictor have quite different structures for the intermediate and end values, which will be shown by the following illustration (Figure 1).

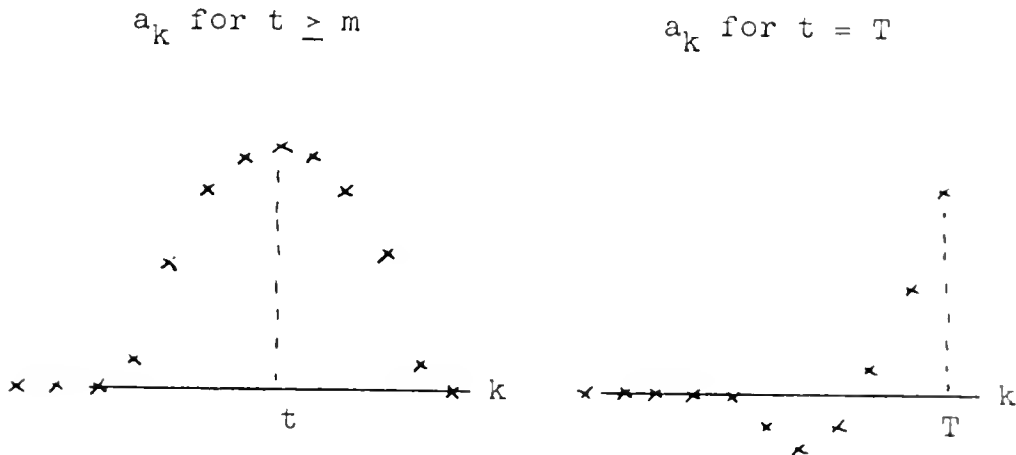


Figure 1

And the case for  $t$  near the end is much more complicated and it is difficult to represent a general solution in a systematic way.

And it must also be remarked that the solution in this case is not equal to the weighted least square estimator with weight proportional to  $|t - k|$ , since for weighted least square estimator  $a_k$  must be of the form

$$a_k = |t - k|(\alpha + \beta_k)$$

which is not the case for intermediate  $k$ .

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13 ABSTRACT Suppose that $Y_i = f(x_i) + U_i$ , $i = 1, \dots, n$ . and $E(U_i) = 0$ . $E(U_i^2) = \sigma^2$ , $E(U_i U_j) = 0$ , $i \neq j$ . We want to estimate $f(x_0)$ by a linear function $\hat{f}(x_0) = \sum a_i Y_i$ . Those coefficients $a_i$ which minimize $\sup E(\hat{f}(x_0) - f(x_0))^2$ for some class of $f$ are sought. In two cases where (a) $\left  \frac{f'(x_i) - f'(x_j)}{x_i - x_j} \right  \leq c\sigma$ , (b) $x_i = i$ , $x_0 = 0$ $ \Delta^{h+1} f(k)  \leq c\sigma$ where $\Delta f(k) = f(k+1) - f(k)$ , the minimax solution is obtained, and it is shown that the solution coincides with weight least squares with weight $w_i = \max(\lambda - u x_i - x_j , 0)$ .			

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